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Polygonal Numbers and Common Differences

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Abstract

The polygonal numbers, or lattices of objects arranged in polygonal shapes, are more than just geometric shapes. The lattices for a polygon with a certain number of sides represent the polygon where the sides increase from length 1, then 2, then 3 and so forth. The mathematical richness emerges when the number of objects in each lattice is counted for the polygons of increasing larger size and then analyzed. So where are the differences? The polygonal numbers being considered in this project represent polygons that have different numbers of sides. It is when the integer powers of the polygonal sequence for a polygon with a specific number of sides are computed and their differences computed, some interesting commonalities emerge. That there are mathematical patterns in the commonalities that arise from differences are both unexpected and pleasing.

Background

A polygonal number is a number represented as dots or pebbles arranged in the shape of a regular polygon, or a polygon where the lengths of all sides are congruent. Consider a triangle as one polygon, and then fill the shape up according to figure 1, the sequence of numbers will form Triangular Numbers. The n^{th} Triangular Numbers is also known as the sum of the first n positive integers. The same can be done with a square as illustrated in figure 2. Using the same visual idea, the Polygonal Numbers with more sides such as Pentagonal Numbers and Hexagonal Numbers can be found.

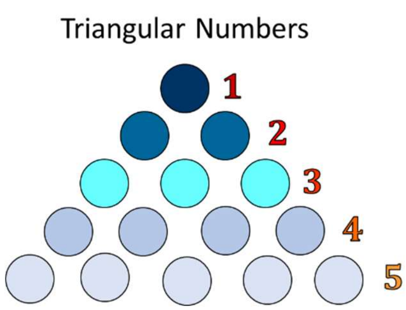

Figure 1

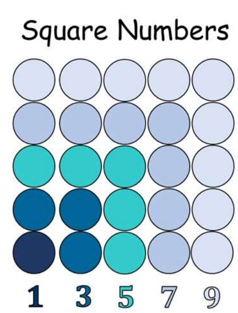

Figure 2

Raising a sequence of Polygonal Numbers to a positive integer power expands the sequence to higher dimensions: an exponent of 2 will result in a lattice in the shape of squares; an exponent of 3 will result in a lattice in the shape of a cube; (The best way of visualizing this phenomenon is to consider the construction of a 2 X 2 X 2 cube from 8 smaller cubes or the construction of a 3 X 3 X 3 cube from 27 smaller units of cube (Figure 3.))

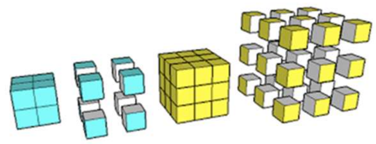

Figure 3: *It should be noted that an exponent of 4 will result in a lattice shape in the shape of a hypercube.*

A curious observation of the differences of the Triangular Numbers when they are each raised to the second power. By taking the difference of second power of any two consecutive numbers in the sequence of Triangular Numbers, the results will be a cubic number. For example, by taking 1 and 3 which are the first and second Triangular Numbers, and finding the difference of their squares (3^2 - 1^2), the results will be 8, which is a cubic number (2^3.) Similarly, by taking 10 and 6 as two other consecutive Triangular numbers and performing the same process, the result will be $10^2 – 6^2$ which is 64 or 4^3. This relationship between differences of the squares of consecutive members sequence brought up a more general question: are there any other relationships between the differences in squares of Polygonal Numbers?

The squares of the Triangular Numbers result from raising every element of the sequence to the power of 2. In other words, the sequence of 1, 3, 6, 10, 15, 21… will transform to 1^2, 3^2, 6^2, 10^2, 15^2, 21^2 … or 1, 9, 36, 100, 255, 441… The same procedure can be used to transform any polygonal sequence of numbers. Table 1 illustrates the numerical value of square, cube and hypercube of Triangular numbers.

Triangular Numbers	**Power of 2**	**Square Numbers**	**Triangular Numbers**	**Power of 3**	**Cubic Numbers**	**Triangular Numbers**	**Power of 4**	**Hyper Cube Numbers**
1	**1**		**1**	**1**		**1**	**1**	
3	**9**		**3**	**27**		**3**	**81**	
6	**36**		**6**	**216**		**6**	**1296**	
10	**100**		**10**	**1000**		**10**	**10000**	
15	**225**		**15**	**3375**		**15**	**50625**	
21	**441**		**21**	**9261**		**21**	**194481**	

Table 1: *Numerical Display of powers of Triangular Numbers*

Common Differences are the differences between successive members of a sequence of numbers, such as the Triangular Numbers seen in Table 2. The first difference represents the actual differences of consecutive members of the sequence. The second difference represents the differences of consecutive first differences. Table 2 shows a more visual illustration of the common differences in Triangular Numbers.

Triangular Numbers	1	**3**	**6**	**10**	**15**
First Difference	3 – 1 = 2	6 – 3 = 3	10 – 6 = 4	15 – 10 = 5	
Second Difference	3 – 2 = 1	4 – 3 = 1	5 – 4 = 1		

Table 2: *First and Second Differences of Triangular Numbers*

Initial research findings

One of the initial observations in Table 1 is that the differences of the squares of consecutive triangular number are a cubic numbers. Table 3 shows the first differences of the squares of the consecutive triangular numbers along with the second and third differences. The third differences introduce an interesting pattern; the third differences are all multiples of six. Table 4 shows this same process applied to the squares of the square numbers while Table 5 shows the same process applied to the squares of the pentagonal numbers.

First Difference	9-1= 8	36-9= 27	100-36= 64	225-100= 125	441-225= 216
Second Difference	27-8=19	64-27= 37	125-64= 61	216-125= 91	
Thirds Difference	37-19= 18	61-37= 24	91-61= 30		
	6*3 = 6*1*3	6*4 = 6*1*4	6*5= 6*1*5		

Table 3: *Differences of Triangular Numbers Raised to the 2^{nd} Power*

First Difference	16-1= 15	81-16= 65	256-81= 175	625-256= 369	1296-625= 671
Second Difference	65-15= 50	175-65= 110	369-175= 194	671-369= 302	
Thirds Difference	110-50= 60	194-110= 84	302-194= 108	434-302= 132	
	6*10 or 6*2*5	6*14 or 6*2*7	6*18 or 6*2*9	6*22 or 6*2*11	

Table 4: *Differences of Square Numbers Raised to the 2nd Power*

First Difference	25-1= 24	144-25= 119	484-144= 340	1225-484= 741	2601-1225= 1376
Second Difference	119-24= 95	340-119= 221	741-340= 401	1376-741= 635	
Third Difference	221-95= 126	401-221= 180	635-401= 234	923-635= 88	
	6*21 or 6*3*7	6*30 or 6*3*10	6*39 or 6*3*13	6*48 or 6*3*16	

Table 5: *Differences of Pentagonal Numbers Raised to the 2nd Power*

Mathematical Proof for Differences of Triangular Numbers Raised to the 2nd Power

The conjecture is that the third differences of the Triangular Numbers produces a multiple of six. The following proof shows this. The general formula for Triangular Numbers is $x\frac{(x+1)}{2}$, therefore the next Triangular Number is $(x+1)\frac{(x+2)}{2}$, and the one after that is $(x+2)\frac{(x+3)}{2}$, $(x+3)\frac{(x+4)}{2}$, etc. where x is a Natural Number. This leads to the following differences.

$$1st\ Diff. \begin{cases} \text{I.} \left((x+1)\dfrac{(x+2)}{2} \right)^2 - \left(x\dfrac{(x+1)}{2} \right)^2 = x^3 + 3x^2 + 3x + 1 \\ \text{II.} \left((x+2)\dfrac{(x+3)}{2} \right)^2 - \left((x+1)\dfrac{(x+2)}{2} \right)^2 = x^3 + 6x^2 + 12x + 8 \\ III. \left((x+3)\dfrac{(x+4)}{2} \right)^2 - \left((x+2)\dfrac{(x+3)}{2} \right)^2 = x^3 + 9x^2 + 27x + 27 \end{cases}$$

$$2nd\ Diff. \begin{cases} IV = II - I = (x^3 + 6x^2 + 12x + 8) - (x^3 + 3x^2 + 3x + 1) \\ \qquad = (3x^2 + 9x + 7) \\ V = III - II = (x^3 + 9x^2 + 27x + 27) - (x^3 + 6x^2 + 12x + 8) \\ \qquad = (3x^2 + 15x + 19) \end{cases}$$

$$3rd\ Diff. V - IV = (3x^2 + 15x + 19) - (3x^2 + 9x + 7) = 6x - 12 = 6(x - 2)$$

Therefore the third difference of squares of Triangular Numbers is a multiple of 6.

Further analysis shows that the second difference has a general form: $(3x^2 + 9x + 7) = (3x^2 + 9x + 6) + 1 = 3(x + 1)(x + 2) + 1$. Since $(x + 1)$ and $(x + 2)$ are two consecutive numbers, they can be written as a multiple of two or $2N$, where N is a Natural Number. This can be written as $3 * 2j + 1 = 6j + 1$, where *j* a is a Natural Number. This shows that the second different of square of the Triangular Numbers is always one more than a multiple of 6.

Mathematical Proof for Differences of Square Numbers Raised to the 2nd Power

The conjecture is that the third differences of the Square Numbers produces a multiple of six. The following proof shows this. The general formula for Square Numbers is x^2, therefore the next Square Number is $(x + 1)^2$, and the one after that is $(x + 2)^2$, $(x + 3)^2$ etc. where *x* is a Natural Number.

$$1st\ \ Diff. \begin{cases} \text{I.} ((x+1)^2)^2 - (x^2)^2 = 4x^3 + 6x^2 + 4x + 1 \\ \text{II.} ((x+2)^2)^2 - ((x+1)^2)^2 = 4x^3 + 18x^2 + 28x + 15 \\ III. ((x+3)^2)^2 - ((x+2)^2)^2 = 4x^3 + 30x^2 + 76x + 65 \end{cases}$$

$$2nd\ Diff. \begin{cases} IV = II - I = (4x^3 + 18x^2 + 28x + 15) - (4x^3 + 6x^2 + 4x + 1) \\ \qquad = (12x^2 + 24x + 14) \\ IV = II - I = (4x^3 + 18x^2 + 28x + 15) - (4x^3 + 6x^2 + 4x + 1) = \\ \qquad = (12x^2 + 48x + 50) \end{cases}$$

$$3rd\ Diff. V - IV = (12x^2 + 48x + 50) - (12x^2 + 24x + 14) = 24x - 36 = 6(4x - 6)$$

Therefore the third difference of squares of Square Numbers is divisible by 6.

Further analysis shows that the second difference has a general form: $(12x^2 + 24\,x + 14) = (12x^2 + 24\,x + 12) + 2 = 6(2x^2 + 4\,x + 2) + 2 = 6j + 2$, where j is a Natural Number. Therefore the second different of square of Square Numbers is always two less than a multiple of 6.

Mathematical Proof for Differences of Pentagonal Numbers Raised to the 2nd Power

The conjecture is that the third differences of the Square Numbers produces a multiple of six. The following proof shows this. The general formula for Pentagonal Numbers is $x\frac{(3x-1)}{2}$. Therefore the next Pentagonal Number is $(x+1)\frac{(3(x+1)-1)}{2}$, and the one after this is $(x+2)\frac{(3(x+2)-1)}{2}$, and the one after this is $(x+3)\frac{(3(x+3)-1)}{2}$ etc. where x is a Natural Number.

$$\begin{cases} \text{I.}\left((x+1)\dfrac{(3x+2)}{2}\right)^2 - \left(x\dfrac{(3x-1)}{2}\right)^2 = 9x^3 + 9x^2 + 5x + 1 \\ \text{II.}\left((x+2)\dfrac{(3x+5)}{2}\right)^2 - \left((x+1)\dfrac{(3x+2)}{2}\right)^2 = 9x^3 + 36x^2 + 50x + 24 \\ III.\left((x+3)\dfrac{(3x+8)}{2}\right)^2 - \left((x+2)\dfrac{(3x+5)}{2}\right)^2 = 9x^3 + 63x^2 + 149x + 119 \end{cases}$$

$$\begin{cases} IV = II - I = (9x^3 + 36x^2 + 50x + 24) - (9x^3 + 9x^2 + 5x + 1) \\ \qquad = (27x^2 + 45x + 23) \\ V = III - II = (9x^3 + 63x^2 + 149x + 119) - (9x^3 + 36x^2 + 50x + 24) \\ \qquad = (27x^2 + 99x + 95) \end{cases}$$

$$V - IV = (27x^2 + 99x + 95) - (27x^2 + 45x + 23) = 54x - 72 = 6(9x - 12)$$

Therefore the third difference of squares of Pentagonal Numbers is a multiple of 6.

Further analysis shows that the second difference has a general form: $(27x^2 + 45x + 23) = (27x^2 + 45x + 24) - 1 = 3(9x^2 + 15x + 8) - 1$. Since all Natural numbers can be represented by either $x = 2N$ or $x = 2N - 1$, the results of this substitution into $9x^2 + 15x + 8$ will be either:

$$\begin{cases} if\ x = 2N,\ 9x^2 + 15x + 8 = 36N^2 + 30N + 8 \\ if\ x = 2N + 1, 9x^2 + 15x + 8 = 9(2N+1)^2 + 15(2N+1) + 8 = 36N^2 + 66N + 32 \end{cases}$$

Continuing, this will yield: $\begin{cases} = 2(18N^2 + 15N + 4) = 2M \\ = 2(18N^2 + 33N + 16) = 2M' \end{cases}$

This shows the second difference of squares of Pentagonal Numbers is:

$\begin{cases} 3 * 2M - 1 = 6M - 1 \\ 3 * 2M' - 1 = 6M' - 1 \end{cases}$ which in either case will be one less than a multiple of 6.

Further Findings

By concentrating more on the results of the third differences in each of the polygonal numbers, other interesting relationships can be seen. The third differences in squares of Triangular Numbers show the following pattern: 6 * 1 * 3, 6 * 1 * 4, 6 * 1 * 5, 6 * 1 * 6 … There is a consistent pattern of 6 * 1, but the third number in the generating expression increases by 1 each time starting with 3. This is illustrated in last line of Table 3.

The third differences of squares of Square Numbers also have a very similar pattern: 6 * 2 * 5, 6 * 2 * 7, 6 * 2 * 9, 6 * 2 * 11… There is a consistent pattern of 6 X 2 but the third numbers in the generating expression increases by 2 each time starting with 5. This is illustrated in the last line of Table 4.

Similarly, the third differences of the squares of the Pentagonal Numbers can be written as 6 * 3 * 7, 6 * 3 * 10, 6 * 3 * 13, 6 * 3 * 16… There is a consistent pattern of 6 * 3 but the third numbers in the generating expression increases by 3 each time starting with 7.

Future research

In the future, to mathematically prove the relationship between the third differences of the squares of Polygonal numbers, the third differences of the general formula for consecutive Polygonal Numbers would need to be shown to produce a multiple of 6.

In addition, difference of higher powers of Polygonal Numbers could be considered. For example, the following proof shows that the third differences of Triangular Numbers raised to the third power produce multiples of 6.

Mathematical Proof for Differences of Triangular Numbers Raised to the 3rd Power

$$1st\ Diff.\begin{cases} \text{I.}\left((x+1)\dfrac{(x+2)}{2}\right)^3 - \left(x\dfrac{(x+1)}{2}\right)^3 = \dfrac{(x+1)^3(3x^2 + 6x + 4)}{4} \\ \text{II.}\left((x+2)\dfrac{(x+3)}{2}\right)^3 - \left((x+1)\dfrac{(x+2)}{2}\right)^3 = \dfrac{(x+2)^3(3x^2 + 12x + 13)}{4} \\ III.\left((x+3)\dfrac{(x+4)}{2}\right)^3 - \left((x+2)\dfrac{(x+3)}{2}\right)^3 = \dfrac{(x+3)^3(3x^2 + 18x + 28)}{4} \end{cases}$$

$$2nd\ Diff.\begin{cases} IV = II - I = \left(\dfrac{(x+2)^3(3x^2+12x+13)}{4}\right) - \left(\dfrac{(x+1)^3(3x^2+6x+4)}{4}\right) \\ V = III - II = \left(\dfrac{(x+3)^3(3x^2+\ 18x+28)}{4}\right) - \left(\dfrac{(x+2)^3(3x^2+12x13)}{4}\right) \end{cases}$$

$$3rd\ Diff.\begin{cases} = \ \left(\dfrac{15x^4+213x^2}{4}+\dfrac{45x^3+117x}{2}+25\right) \\ = \left(\dfrac{15x^4+573x^2}{4}+\dfrac{75x^3+495x}{2}+163\right) \end{cases}$$

$$V - IV = \left(\frac{15x^4+573x^2}{4}+\frac{75x^3+495x}{2}+163\right) - \left(\frac{15x^4+213x^2}{4}+\frac{45x^3+117x}{2}+25\right)$$
$$= 15x^3+90x^2+189x+138 = 3(5x^3+30x^2+63x+46)$$

By considering the fact that every Natural Number can be written as $2N$ or $2N+1$, the expression in parenthesis will be:

$$\begin{cases} if\ x\ =\ 2N,\ 5x^3+30x^2+63x+46\ =\ 40N^3+120N^2+126N+46 \\ if\ x = 2N+1, 5x^3+30x^2+63x+46 = 5(2N+1)^3+30(2N+1)^3+63(2N+1)+46 \end{cases}$$

$$\begin{cases} = 2(20N^3+60N^2+63N+23\ ) =\ 2M \\ = 40N^3+180N^2+276N+144 = 2(20N^3+90N^2+138N+72)\ =\ 2M' \end{cases}$$

As the results, the third difference of cubes of Triangular Numbers is: $\begin{cases} 3*2M = 6M-1 \\ 3*2M' = 6M'-1 \end{cases}$ that in either case, it will be divisible by 6

Summary

In summary, it had been shown that the third differences in squares of Polygonal Numbers follow a very specific algorithm. The third differences of the squares of the Triangular Numbers can be written as 6 * 1 * (n + 2) where n is a Natural Number. The third differences of the squares of Square Numbers can be written as 6 * 2 * (2n + 3) where n is a Natural Number. The third differences of the squares of the Pentagonal Numbers can be written as 6 * 3 * (3n + 4) where n is a natural number. So in general, the formula for each Polygonal is 6 * k * (k * n + k+1) where “k” is two less than the number of sides in the polygon.

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